

On Simultaneous Chebyshev Approximation

G. M. PHILLIPS AND J. H. MCCABE

Mathematics Department, University of St. Andrews, St. Andrews, Scotland

AND

E. W. CHENEY*

Mathematics Department, University of Texas, Austin, Texas 78712

Communicated by Oved Shisha

Received March 27, 1978

1. INTRODUCTION

Let f_1, \dots, f_n belong to $C[a, b]$, and let G be a nonempty subset of $C[a, b]$. The problem addressed here is that of approximating the n functions f_i simultaneously, in some sense, by an element of G .

This problem and variations on it have attracted much interest, and we have included a complete list of references known to us. Not included in the references are works concerning the related problem of *widths* of function classes, nor works concerned with approximating one function and its derivatives.

A number of natural definitions of a simultaneous approximant may be offered. For example, Dunham [4] and Diaz and McLaughlin [6] investigated the problem of selecting $g \in G$ to minimize the expression

$$\max_{1 \leq i \leq n} \max_{a \leq x \leq b} |f_i(x) - g(x)|.$$

Ling, in [19], considered (with $n = 2$) the problem of minimizing the expression $\max_{a \leq x \leq b} \sum_{i=1}^n |f_i(x) - g(x)|$. This is a special case of a more general situation in which a not-necessarily-finite number of functions f_i is given, and a summation or integration process is involved in the error criterion. It is this type of simultaneous approximation that is investigated here.

* Supported by Grant DAAG29-77-G-0215 from the U.S. Army Research Office.

2. THE $\mathcal{L}_1 - \mathcal{L}_\infty$ SIMULTANEOUS APPROXIMATION

The finite set of univariate functions in Section 1 is now replaced by a single bivariate function $f(x, y)$. In order for everything to work smoothly we assume that x ranges over a compact Hausdorff space X , and y ranges over a measure space Y of measure 1. Furthermore, $f(x, y)$ is continuous in x for fixed y and integrable in y for fixed x . The *mean* of f is defined to be

$$\bar{f}(x) = \int_Y f(x, y) dy \quad (1)$$

and the supremum norm of an element g of $C(X)$ is defined by $\|g\|_\infty = \max_x |g(x)|$.

An element g^* in a prescribed class $G \subset C(X)$ is now termed a *best simultaneous approximation* of f if for all $g \in G$,

$$\max_x \int |f(x, y) - g^*(x)| dy \leq \max_x \int |f(x, y) - g(x)| dy.$$

The finite case is recovered, of course, by taking Y to be a measure space with only n points, each of mass $1/n$. But there are important cases, notably in the approximation of integral transforms, when the full generality of this setting is required.

LEMMA. *If $g \in C(X)$ and $|\bar{f}(x) - f(x, y)| \leq \|\bar{f} - g\|_\infty$ for all x and y then $\|\bar{f} - g\|_\infty = \max_x \int |f(x, y) - g(x)| dy$.*

Proof. Let v be any element of $\mathcal{L}_\infty(Y)$ satisfying the equation $|v(y)| = 1$ almost everywhere. Let A be the set of y such that $v(y) = 1$, and B the set of y such that $v(y) = -1$. Then for all x ,

$$\begin{aligned} & \int v(y)[f(x, y) - g(x)] dy \\ &= \int_A [f(x, y) - g(x)] dy + \int_B [g(x) - f(x, y)] dy \\ &= \int_Y f(x, y) dy - \int_B f(x, y) dy - g(x) \mu(A) + g(x) \mu(B) - \int_B f(x, y) dy \\ &= \bar{f}(x) - 2 \int_B f(x, y) dy - g(x)[1 - 2\mu(B)] \\ &= 2 \int_B [\bar{f}(x) - f(x, y)] dy + [\bar{f}(x) - g(x)][1 - 2\mu(B)]. \end{aligned}$$

Here the equation $\mu(A) + \mu(B) = 1$ was used. If $\mu(B) \leq \frac{1}{2}$ then the preceding equation yields

$$\int v(y)[f(x, y) - g(x)] dx \leq 2\mu(B) \|\bar{f} - g\|_\infty + \|\bar{f} - g\|_\infty [1 - 2\mu(B)] = \|\bar{f} - g\|_\infty.$$

If $\mu(B) > \frac{1}{2}$ then a similar line of reasoning leads to the same inequality with A in place of B . Hence we may take first a supremum for all v and then a supremum for all x to conclude that

$$\begin{aligned} \max_x \int |f(x, y) - g(x)| dy &\leq \|\bar{f} - g\|_\infty = \max_x \left| \int [f(x, y) - g(x)] dy \right| \\ &\leq \max_x \int |f(x, y) - g(x)| dy \quad \blacksquare \end{aligned}$$

THEOREM. *If $g^* \in G$ and if for all $g \in G$ and $y \in Y$ $\max_x |\bar{f}(x) - f(x, y)| \leq \max_x |\bar{f}(x) - g^*(x)| \leq \max_x |\bar{f}(x) - g(x)|$ then $\max_x \int |f(x, y) - g^*(x)| dy \leq \max_x \int |f(x, y) - g(x)| dy$.*

The theorem follows at once from the lemma. It states that if the functions $f(\cdot, y)$ are closer to their mean, \bar{f} , than \bar{f} is to G , then the best simultaneous approximation is nothing but the best approximation to \bar{f} in the usual sense. In many applications, the functions in the family are close to one another and the hypothesis is fulfilled.

An interesting special case is that in which Y is a set of n points, each of mass $1/n$, and X is an interval $[a, b]$ on the real line. Ling [19] proves the above theorem in this setting for the case $n = 2$.

EXAMPLE 1. Take X and Y to be unit intervals $[0, 1]$, and let G be π_1 , the set of linear polynomials. Let

$$f(x, y) = x^{1/2}(1 + \epsilon \sin 2\pi y).$$

It is easily verified that the hypotheses of the above theorem hold when $|\epsilon| \leq \frac{1}{8}$ and $g^*(x) = x + \frac{1}{8}$. Thus g^* is the best simultaneous approximation to $f(x, y)$.

An example showing that the theorem does not encompass all cases is as follows.

EXAMPLE 2. Let X be a two-point space, $X = \{p, q\}$, and Y a five-point space, $Y = \{1, 2, \dots, 5\}$. Let f be given by the table

x	p	q	p	q	p	q	p	q	p	q
y	1	1	2	2	3	3	4	4	5	5
$f(x, y)$	1	0	0	1	1	1	-1	0	0	-1

Let G consist of all functions g on X such that $g(p) = g(q)$. Then $\bar{f} = (.2, .2) \in G$, while the best simultaneous approximation of $f(x, y)$ is 0.

3. CHARACTERIZATION OF BEST SIMULTANEOUS APPROXIMATION

In this section, several mild hypotheses are imposed upon the simultaneous approximation problem of Section 2 in order that a characterization can be established for best approximations.

It is assumed that $f: X \times Y \rightarrow \mathbb{R}$, where X is a compact topological space and Y is a measure space. Furthermore, these conditions are imposed:

- (1) $\{f(\cdot, y): y \in Y\}$ is an equicontinuous family of functions from X to \mathbb{R} .
- (2) For each $x \in X$, $f(x, \cdot)$ is an integrable function from Y to \mathbb{R} .
- (3) For each x , the zeros of $f(x, \cdot)$ form a set of measure zero in Y .

As in Section 2, a simultaneous approximation of the family $\{f(\cdot, y): y \in Y\}$ is sought, by an element g of a prescribed subset G in $C(X)$. It is assumed now that the set G is convex. An element of $g \in G$ is a best simultaneous approximation to f if it minimizes the expression

$$\sup_{x \in X} \int_Y |f(x, y) - g(x)| dy.$$

THEOREM. *In order that a particular element $g^* \in G$ shall minimize the above expression, it is necessary and sufficient that each element $g \in G$ shall satisfy the inequality*

$$\inf_{x \in K} [g(x) - g^*(x)] \int_Y \text{sgn}[f(x, y) - g^*(x)] dy \leq 0,$$

where K is the critical set, i.e., the set of x in X for which

$$\int_Y |f(x, y) - g^*(x)| dy = \sup_t \int_Y |f(t, y) - g^*(t)| dy.$$

Proof. This follows from the main theorem of [29], if the family \mathcal{F} of

that paper is taken to be the set of all functions $f(x, y) - g(x)$ as g ranges over G . ■

A special case of the theorem arises when X is an interval and G is an n -dimensional Haar subspace in $C(X)$. In this case, an element $g^* \in G$ is a best simultaneous approximation of f if and only if the function

$$\phi(x) \equiv \int_Y \operatorname{sgn}[f(x, y) - g^*(x)] dy$$

alternates in sign $n + 1$ times on the critical set. That is, there should exist points $x_0 < x_1 < \dots < x_n$ in X such that

$$\int_Y |f(x_i, y) - g^*(x_i)| dy = \sup_t \int_Y |f(t, y) - g^*(t)| dy \quad (1)$$

and

$$\phi(x_i) \phi(x_{i-1}) < 0 \quad (1 \leq i \leq n). \quad (2)$$

REFERENCES

1. E. REMES, Sur la détermination des polynômes d'approximation de degré donné, *Comm. Soc. Math. Kharkov Ser. 4* **10** (1934), 41-63.
2. M. GOLOMB, "On the Uniformly Best Approximation of Functions Given by Incomplete Data," Report 121, Math. Research Center, University of Wisconsin, December 1959.
3. M. GOLOMB, "Lectures on Theory of Approximation," pp. 264 ff., Argonne National Laboratory, 1962.
4. C. B. DUNHAM, Simultaneous Chebyshev approximation of functions on an interval, *Proc. Amer. Math. Soc.* **18** (1967), 472-477.
5. E. BREDENDIEK, Simultanapproximationen, *Arch. Rational Mech. Anal.* **33**(4) (1969), 307-330.
6. J. B. DIAZ AND H. W. McLAUGHLIN, Simultaneous approximation of a set of bounded real functions, *Math. Comp.* **23** (1969), 538-594.
7. J. B. DIAZ AND H. W. McLAUGHLIN, Simultaneous Chebyshev approximation of a set of bounded complex-valued functions, *J. Approximation Theory* **6** (1969), 419-432.
8. P. J. LAURENT AND P. D. TUAN, Global approximation of a compact set by elements of a convex set in a normed space, *Numer. Math.* **15** (1970), 137-150.
9. H. P. BLATT, "Nicht-Lineare Gleichmässige Simultanapproximation," Dissertation, Universität des Saarlandes, Saarbrücken, 1970.
10. M. P. CARROLL, Simultaneous L_1 approximation of a compact set of real valued functions, *Numer. Math.* **19** (1972), 110-115.
11. J. B. DIAZ AND H. W. McLAUGHLIN, On simultaneous Chebyshev approximation and Chebyshev approximation with an additive weight, *J. Approximation Theory* **6** (1972), 68-71.
12. H.-P. BLATT, Nicht-lineare gleichmässige Simultanapproximation, *J. Approximation Theory* **8** (1973), 210-248.
13. H.-P. BLATT, Über rationale Tschebyscheff-Approximation mehrerer Funktionen, *J. Approximation Theory* **9** (1973), 126-148.

14. M. P. CARROLL, On simultaneous L_1 approximation, in "Approximation Theory" (G. G. Lorentz, Ed.), pp. 295–297, Academic Press, New York, 1973.
15. K.-P. LIM, Simultaneous approximation of compact sets by elements of convex sets in normed linear spaces, *J. Approximation Theory* **12** (1974), 332–351.
16. D. S. GOEL, A. S. B. HOLLAND, C. NASIM, AND B. N. SAHNEY, "Characterization of an Element of Best L^p -Simultaneous Approximation," pp. 10–14, S. R. Ramanujan Memorial Volume, Madras, 1974.
17. D. S. GOEL, A. S. B. HOLLAND, C. NASIM, AND B. N. SAHNEY, On best simultaneous approximation in normed linear spaces, *Canad. Math. Bull.* **17** (1974), 523–527.
18. F. DEUTSCH, Review of [17], *Math. Reviews* **51**, No. 13556.
19. W. H. LING, On simultaneous Chebyshev approximation in the sum norm, *Proc. Amer. Math. Soc.* **48** (1975), 185–188.
20. A. BRØNDSTED, A note on best simultaneous approximation in normed linear spaces, *Canad. Math. Bull.* **19** (1976), 359–360.
21. A. S. B. HOLLAND, B. N. SAHNEY, AND J. TZIMBALARIO, On best simultaneous approximation, *J. Indian Math. Soc.* **40** (1976), 69–73.
22. G. M. PHILLIPS AND B. N. SAHNEY, Best simultaneous approximation in the L_1 and L_2 norms, in "Theory of Approximation with Applications" (A. G. Law and B. N. Sahney, Eds.), pp. 213–219, Academic Press, New York, 1976.
23. A. S. B. HOLLAND AND B. N. SAHNEY, Some remarks on best simultaneous approximation, in "Theory of Approximation with Applications" (A. G. Law and B. N. Sahney, Eds.), pp. 332–337, Academic Press, New York, 1976.
24. M. J. GILLOTTE AND H. W. McLAUGHLIN, On additive weight approximation and simultaneous Chebyshev approximation using varisolvent families, *J. Approximation Theory* **17** (1976), 35–43.
25. A. S. B. HOLLAND, B. N. SAHNEY, AND J. TZIMBALARIO, On best simultaneous approximation, *J. Approximation Theory* **17** (1976), 187–188.
26. P. D. MILMAN, On best simultaneous approximation in normed linear spaces, *J. Approximation Theory* **20** (1977), 223–238.
27. W. H. LING AND H. W. McLAUGHLIN, Approximation of random functions, *J. Approximation Theory* **20** (1977), 10–22.
28. C. B. DUNHAM, Simultaneous approximation of a set, *J. Approximation Theory* **21** (1977), 205–208.
29. G. M. PHILLIPS, J. H. MCCABE, E. W. CHENEY, A mixed-norm bivariate approximation problem with applications to Lewanowicz operators, in "Proceedings of a Conference on Approximation Theory" (D. C. Handscomb, Ed.), pp. 315–323, Academic Press, New York, 1978.
30. A. S. B. HOLLAND, J. H. MCCABE, G. M. PHILLIPS, AND B. N. SAHNEY, Best simultaneous L_1 approximations, *J. Approximation Theory* **24** (1978), 361–365.